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Linear Algebra and its Applications 423 (2007) 90–98
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Spectral results on graphs with regularity constraints

Domingos M. Cardoso, Paula Rama *

Dep. de Matemática, Univ. Aveiro, 3810-193 Aveiro, Portugal

Received 14 July 2006; accepted 28 September 2006
Available online 28 November 2006
Submitted by D. Cvetković

Abstract

Graphs with (k, τ)-regular sets and equitable partitions are examples of graphs with regularity constraints. A (k, τ)-regular set of a graph G is a subset of vertices $S \subseteq V(G)$ inducing a k-regular subgraph and such that each vertex not in S has τ neighbors in S. The existence of such structures in a graph provides some information about the eigenvalues and eigenvectors of its adjacency matrix. For example, if a graph G has a (k_1, τ_1)-regular set S_1 and a (k_2, τ_2)-regular set S_2 such that $k_1 - \tau_1 = k_2 - \tau_2 = \lambda$, then λ is an eigenvalue of G with a certain eigenvector. Additionally, considering primitive strongly regular graphs, a necessary and sufficient condition for a particular subset of vertices to be (k, τ)-regular is introduced. Another example comes from the existence of an equitable partition in a graph. If a graph G, has an equitable partition π then its line graph, $L(G)$, also has an equitable partition, $\bar{\pi}$, induced by π, and the adjacency matrix of the quotient graph $L(G)/\bar{\pi}$ is obtained from the adjacency matrix of G/π.

Keywords: Graph theory; Graph spectra; Equitable partitions; Strongly regular graphs

1. Introduction

In this paper we present some properties concerning the eigenvalues and eigenvectors of graphs with (k, τ)-regular sets and equitable partitions. The (k, τ)-regular sets emerged in [11,19] related to strongly regular graphs and designs and later, in the study of graphs with domination constraints [7,18]. Equitable partitions were introduced in [14–16]. In the last of these equitable bipartitions are used to obtain information about eigenvalues and eigenvectors of graphs. Equitable partitions appear related to automorphism groups of graphs [9], coding theory [5], walk partitions [12]

* Corresponding author.
E-mail address: prama@mat.ua.pt (P. Rama).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2006.09.028

and distance-regular graphs (see [6], for details and further references). Recent applications of equitable partitions may be found in [10] applied to the study of graphs with three eigenvalues, in [17] applied to the study of spectral properties of configuration graphs associated to landscapes and in [1] in the characterization of graphs with convex quadratic independent number.

In this paper we consider a simple graph G of order n with a set of vertices $V(G)$ and a set of edges $E(G)$. An element of $E(G)$, which has the vertices i and j as end-vertices, is denoted by ij. If $v \in V(G)$, then we denote the neighborhood of v by $N_G(v)$ ($N_G(v) = \{w : vw \in E(G)\}$) and the closed neighborhood of v is denoted by $N_G[v]$ ($N_G[v] = N_G(v) \cup \{v\}$). The *line graph* $L(G)$ of a graph G has the edges of G as its vertices, with two vertices of $L(G)$ being adjacent if and only if the corresponding edges of G have a vertex in common. Given a subset of vertices S of a graph G, the vector $\mathbf{x} \in \mathbb{R}^V$ with $\mathbf{x}_v = 1$ if $v \in S$ and $\mathbf{x}_v = 0$ if $v \notin S$ is called the *characteristic vector* of S. Given a partition $\pi = (V_1, \ldots, V_r)$ of $V(G)$, the matrix P^π whose columns are the characteristic vectors of the subsets $V_1, \ldots, V_r$, is called the *characteristic matrix* of π. Along this paper, A_G will denote the *adjacency matrix* of the graph G of order $n > 1$, that is, $A_G = (a_{ij})_{n\times n}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } ij \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Note that A_G is a symmetric matrix and thus has n real eigenvalues. The *spectrum* of a matrix A (that is, the multi-set of its eigenvalues) will be denoted by $\sigma(A)$, and the null space by $\mathrm{Ker}(A)$. The *multiplicity of an eigenvalue* $\lambda \in \sigma(A_G)$ is the multiplicity of λ as a zero of the characteristic polynomial of A_G. Throughout the text $\hat{\mathbf{e}}$ will denote the all-ones vector with n components and I will denote the identity matrix.

The paper is organized as follows. In the next section we present some results on the existence of certain eigenvalues and eigenvectors of graphs with (k, τ)-regular sets. In Section 3 we conclude that the existence of an equitable partition on a graph G allows us to obtain some information not only about the spectrum of the graph G but also on the spectrum of its line graph $L(G)$. Finally, in Section 4, we deduce a necessary and sufficient condition, based on the eigenvectors, for a subset of vertices of a primitive strongly regular graph to be a (k, τ)-regular set.

2. Eigenvalues and eigenvectors corresponding to (k, τ)-regular sets

According to [18] (see also [7]) a (ρ, γ)-*set* of a graph G of order n is a non-empty subset of vertices $S \subseteq V(G)$ such that

$$|N_G(v) \cap S| \in \rho, \;\; \forall v \in S, \quad \text{and} \quad |N_G(v) \cap S| \in \gamma, \;\; \forall v \in V(G) \setminus S,$$

where ρ and γ are non-empty subsets of $\{0, 1, \ldots, n-1\}$. If the subsets ρ and γ have a single number, $\rho = \{k\}$ and $\gamma = \{\tau\}$, S is a (k, τ)-regular set, that is, $S \subseteq V(G)$ is a (k, τ)-regular set if and only if S is a $(\{k\}, \{\tau\})$-set. According to this definition, if G is a p-regular graph then the set of vertices, $V(G)$, is a (p, τ)-regular set for every $\tau \in \mathbb{N}_0$ (with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$). However, by convention, through all the paper the set of vertices $V(G)$ of a p-regular graph G is just a $(p, 0)$-regular set.

From the above definition, we may conclude that a (k, τ)-regular set S of a graph G is a non-empty subset of vertices that induces a k-regular subgraph and such that every vertex not in S has τ neighbors in S. Hence, if G has a (k, τ)-regular set S then S is a $(|S| - k - 1, |S| - \tau)$-regular set of the complement, $\bar{G}$. In addition, if G is p-regular, then $V(G) \setminus S$ is a $(p - \tau, p - k)$-regular set of G. For the graph G, depicted in Fig. 1, $S_1 = \{1, 2, 3\}$ is a $(\{1, 2\}, \{1\})$-set and $S_2 = \{1, 4\}$ is a $(0, 1)$-regular set.

From the next result we conclude that if a regular graph G has a (k, τ)-regular set then $k - \tau \in \sigma(A_G)$. Thus, if G is a p-regular graph (with $p > 0$) and $\sigma(A_G) \cap \mathbb{Z} = \{p\}$ then G has not a (k, τ)-regular set $S \subset V(G)$, with $\tau > 0$.

Proposition 2.1 [19]. *Let G be a p-regular graph and $\mathbf{x}$ the characteristic vector of $S \subseteq V(G)$. Then S is a (k, τ)-regular set of G, with $\tau > 0$, if and only if $k - \tau \in \sigma(A_G)$ with corresponding eigenvector $(p - k + \tau)\mathbf{x} - \tau\hat{\mathbf{e}}$.*

From the definition of (k, τ)-regular set we get the following lemma.

Lemma 2.1 [2]. *A graph G has a (k, τ)-regular set S if and only if the characteristic vector, $\mathbf{x}$, of S is a solution of the linear system*

$$(A_G - (k - \tau)I)\mathbf{x} = \tau\hat{\mathbf{e}}. \tag{1}$$

As a consequence of this lemma, we may say that a graph G has a (k, τ)-regular set if and only if the linear system (1) has a 0–1 solution and, additionally, this solution is the characteristic vector of a (k, τ)-regular set.

The first part of the next proposition was introduced in [13].

Proposition 2.2. *Let $\lambda \in \mathbb{Z}$ and G be a graph with a (k_1, τ_1)-regular set S_1 $(\tau_1 > 0)$ and a (k_2, τ_2)-regular set S_2, such that $S_1 \neq S_2$ and $k_1 - \tau_1 = k_2 - \tau_2 = \lambda$. Then $\lambda \in \sigma(A_G)$ with corresponding eigenvector $\frac{\tau_2}{\tau_1}\mathbf{x}_1 - \mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the characteristic vectors of S_1 and S_2, respectively.*

Proof. Since $S_1 \neq S_2$ and $\mathbf{x}_1, \mathbf{x}_2 \in \{0, 1\}^n$ $(n = |V(G)|)$ then $\mathbf{x}_1 \neq \mathbf{x}_2$ and $\frac{\tau_2}{\tau_1}\mathbf{x}_1 \neq \mathbf{x}_2$. In addition, $\frac{\tau_2}{\tau_1}\mathbf{x}_1$ is a solution of (1) with $(k, \tau) = (k_2, \tau_2)$ concluding that for such values of the parameters k and τ, the system (1) has an infinite number of solutions, that is, $\det(A_G - \lambda I) = 0$ and $\lambda \in \sigma(A_G)$. Finally, taking into account that the system (1) is equivalent to $A_G\mathbf{x} = (k - \tau)\mathbf{x} + \tau\hat{\mathbf{e}}$, it follows that

$$\begin{aligned}
A_G\left(\frac{\tau_2}{\tau_1}\mathbf{x}_1 - \mathbf{x}_2\right) &= \frac{1}{\tau_1}(\tau_2 A_G\mathbf{x}_1 - \tau_1 A_G\mathbf{x}_2) \\
&= \frac{1}{\tau_1}(\tau_2\lambda\mathbf{x}_1 + \tau_2\tau_1\hat{\mathbf{e}} - \tau_1\lambda\mathbf{x}_2 - \tau_1\tau_2\hat{\mathbf{e}}) \\
&= \frac{\lambda}{\tau_1}(\tau_2\mathbf{x}_1 - \tau_1\mathbf{x}_2) = \lambda\left(\frac{\tau_2}{\tau_1}\mathbf{x}_1 - \mathbf{x}_2\right),
\end{aligned}$$

leading to the conclusion that $\frac{\tau_2}{\tau_1}\mathbf{x}_1 - \mathbf{x}_2$ is an eigenvector of A_G corresponding to the eigenvalue λ. $\square$

As an example, the graph G in Fig. 1 has two $(2, 1)$-regular sets ($\{1, 2, 5, 6\}$ and $\{2, 3, 4, 5\}$) and two $(0, 1)$-regular sets ($\{1, 4\}$ and $\{3, 6\}$) so, $\{-1, 1\} \subset \sigma(A_G) = \{-2.41, -1, -0.41, 0.41, 1, 2.41\}$. Furthermore, $[1\ 0\ -1\ 1\ 0\ -1]^\top$ is an eigenvector corresponding to the eigenvalue -1 and $[1\ 0\ -1\ -1\ 0\ 1]^\top$ is an eigenvector associated with the eigenvalue 1. This graph has also a $(1, 1)$-regular set, $\{2, 5\}$, and $0 \notin \sigma(A_G)$ (note that there is no other (k, τ)-regular set $S \subseteq V(G)$ with $k - \tau = 0$). Other examples of (k, τ)-regular sets may be obtained, for instance, from a Hamiltonian graph G, where each Hamilton cycle defines a $(2, 4)$-regular set in $L(G)$ (see

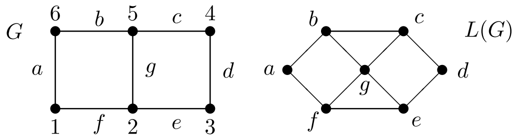

Fig. 1. An equitable bipartition $\pi = (\{1, 2, 3, 4\}, \{2, 5\})$ of G and the corresponding equitable partition $\bar{\pi} = (\{a, d\}, \{b, c, e, f\}, \{g\})$ of $L(G)$.

Fig. 1) and a (2, 2)-regular set in $\widehat{G}$, denoting by $\widehat{G}$ the graph obtained from G by subdivision of all its edges (that is, inserting one new vertex in the middle of each edge). In addition, from Proposition 2.2, we may also conclude that if a Hamiltonian graph has even order then, considering $L(C) \subset V(L(G))$ the subset of vertices of $L(G)$ corresponding to the edges of a Hamiltonian cycle of G and $L(M) \subset L(C)$ the subset of vertices of $L(G)$ corresponding to the edges of a perfect matching $M \subset E(G)$, then $\mathbf{u} \in \mathbb{R}^n$ such that

$$\mathbf{u}_i = \begin{cases} 1 & \text{if } i \in C \setminus M, \\ -1 & \text{if } i \in C \cap M, \\ 0 & \text{otherwise} \end{cases}$$

is an eigenvector of $L(G)$ associated to the eigenvalue -2. A similar conclusion was obtained in [4] using different arguments.

In Proposition 2.2 we consider at least, a (k, τ)-regular set with $\tau > 0$. A $(k, 0)$-regular set of a graph G is the union of the subsets of vertices of one or more connected components of G that induce k-regular subgraphs. Thus, if G has a $(k, 0)$-regular set then $k \in \sigma(A_G)$ and the characteristic vector of S is an eigenvector associated to the eigenvalue k.

3. Graphs with equitable partitions

According to [16], a partition $\pi = (V_1, V_2, \ldots, V_r)$ of $V(G)$ is *equitable* if for all $i, j \in \{1, 2, \ldots, r\}$ there exists $d_{ij} \in \mathbb{N}_0$ such that

$$\forall v \in V_i, \quad |N_G(v) \cap V_j| = d_{ij}, \tag{2}$$

that is, the number of neighbors that each vertex of V_i has in V_j is independent of the choice of the vertex in V_i. If $r = 2$ in (2), we say that π is an *equitable bipartition*. From now on, for simplicity of language, an equitable partition of the vertex set $V(G)$ of a graph G will also be called an equitable partition of the graph G.

Let G be a graph with an equitable partition $\pi = (V_1, \ldots, V_r)$. According to [6], the *quotient graph* G/π of G with respect to the equitable partition π is a multi-digraph with the subsets of π as its vertices and with d_{ij} arcs going from V_i to V_j. Thus G/π has, in general, loops and multiple arcs. The adjacency matrix $A_{G/\pi}$ is a $r \times r$ matrix with entries d_{ij}. In particular, if G is a p-regular graph and $S \subset V(G)$ is a (k, τ)-regular set then $\pi = (S, V(G) \setminus S)$ is an equitable bipartition such that

$$A_{G/\pi} = \begin{bmatrix} k & p-k \\ \tau & p-\tau \end{bmatrix}.$$

As a consequence, from the above definitions, a bipartition $\pi = (S, V(G) \setminus S)$ is equitable with

$$A_{G/\pi} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

if and only if S is (d_{11}, d_{21})-regular and $V(G) \setminus S$ is (d_{22}, d_{12})-regular.

From the next result we conclude that the multiplicity of λ as an eigenvalue of A_G is at least its multiplicity as an eigenvalue of $A_{G/\pi}$, that is, taking into account the multiplicities of the eigenvalues, we have $\sigma(A_{G/\pi}) \subset \sigma(A_G)$.

Proposition 3.1 [8]. *If π is an equitable partition of a graph G then $P_{A_{G/\pi}}(\lambda)$ divides $P_{A_{G(\lambda)}}$, where $P_A(\lambda)$ denotes the characteristic polynomial of the matrix A.*

In the following proposition we prove that if π is an equitable partition of a graph G then, it is possible to obtain an equitable partition of $L(G)$ from π.

Proposition 3.2 [13]. *If G is a graph with an equitable partition $\pi = (V_1, \ldots, V_r)$, with $A_{G/\pi} = [d_{ij}]$, then $L(G)$ has an equitable partition $\bar{\pi} = \{U_{ij} : d_{ij} > 0$ and $1 \leqslant i \leqslant j \leqslant r\}$, where U_{ij} denotes the subset of vertices of $L(G)$ corresponding to the subset of edges $u = xy \in E(G)$ such that $x \in V_i$ and $y \in V_j$. The number of subsets of $\bar{\pi}$ is at most $\frac{r+r^2}{2}$.*

Proof. Consider $i, j, k, l \in \{1, 2, \ldots, r\}$ with $i \leqslant j, k \leqslant l$. Let $v = xy \in E(G)$, with $x \in V_i$ and $y \in V_j$. We note that the number of edges of $E(G)$, corresponding to vertices of $U_{kl} \subset V(L(G))$ that are adjacent to v does not depend on the choice of v in the subset of edges of $E(G)$ incident in one vertex of V_i and a vertex of V_j, being constant for every choice of i, j, k, l. In fact, $\forall v \in U_{ij} \subset V(L(G))$

$$|N_{L(G)}(v) \cap U_{kl}| = \begin{cases} 2d_{ii} - 2 & \text{if } i = j = k = l, \\ 2d_{ih} & \text{if } i = j \in \{k, l\},\ k \neq l,\ h \in \{k, l\} \setminus \{i, j\}, \\ d_{kk} & \text{if } k = l \in \{i, j\} \text{ and } i \neq j, \\ d_{kl} & \text{if } k \in \{i, j\} \text{ and } |\{i, j, l\}| = 3, \\ d_{lk} & \text{if } l \in \{i, j\} \text{ and } |\{i, j, k\}| = 3, \\ 0 & \text{if } \{i, j\} \cap \{k, l\} = \emptyset, \\ d_{ij} + d_{ji} - 2 & \text{if } i = k \neq j = l. \end{cases} \tag{3}$$

Finally let us show that the number of subsets of $\bar{\pi}$ is at most $\frac{r+r^2}{2}$. If $d_{ij} > 0$ for all $i, j \in \{1, \ldots, r\}$ then, for each $j \in \{1, \ldots, r\}$, the number of subsets U_{ij} with $i \leqslant j$ is given by j. Hence, the number of subsets of $\bar{\pi}$ is given by $\sum_{j=1}^{r} j = \frac{r+r^2}{2}$. $\square$

The graph G, depicted in Fig. 1, has an equitable partition $\pi = (V_1, V_2)$, where $V_1 = \{1, 3, 4, 6\}$ and $V_2 = \{2, 5\}$, with $A_{G/\pi} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.

According to Proposition 3.2, $L(G)$ has an equitable partition $\bar{\pi} = (U_{11}, U_{12}, U_{22})$, where $U_{11} = \{a, d\}$, $U_{12} = \{b, c, e, f\}$ and $U_{22} = \{g\}$, with

$$A_{L(G)/\bar{\pi}} = \begin{array}{c} \\ U_{11} \\ U_{12} \\ U_{22} \end{array} \begin{array}{c} \begin{array}{ccc} U_{11} & U_{12} & U_{22} \end{array} \\ \begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 4 & 0 \end{pmatrix} \end{array}.$$

As it is well known (see, for example, [4]), if G is a p-regular graph ($p > 0$) and $\lambda \neq -p$ is an eigenvalue of A_G, then $\lambda + p - 2$ is an eigenvalue of $A_{L(G)}$ with the same multiplicity as λ. In Proposition 3.2 it is shown that if G is an arbitrary graph with an equitable partition π then, $L(G)$ has an equitable partition $\bar{\pi}$ with a matrix $A_{L(G)/\bar{\pi}}$ whose entries are described in (3). Consider-

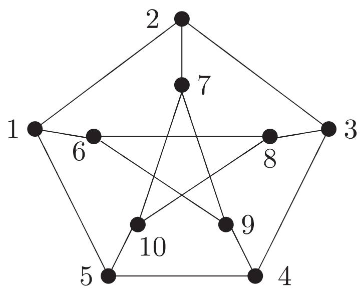

Fig. 2. Petersen graph.

ing now Proposition 3.1, we conclude that $\sigma(A_{L(G)/\bar{\pi}}) \subset \sigma(A_{L(G)})$. Returning to the graphs in Fig. 1, we have

$$\sigma(A_{L(G)/\bar{\pi}}) = \{[-2]^1, [0]^1, [3]^1\} \subset \sigma(A_{L(G)}) = \{[-2]^2, [-1]^1, [0]^1, [1]^2, [3]^1\}^1.$$

From the following result, if a graph G has an equitable partition π, the eigenvectors of $A_{G/\pi}$ corresponding to eigenvalues $\lambda \in \sigma(A_{G/\pi})$ can be extended to be eigenvectors of A_G, while the eigenvectors of A_G associated with eigenvalues $\lambda \in \sigma(A_G) \setminus \sigma(A_{G/\pi})$ are orthogonal to the characteristic vectors of the subsets of π.

Proposition 3.3 [6]. *Let G be a graph of order n with an equitable partition $\pi = (V_1, \ldots, V_r)$ and let P^π be the characteristic matrix of π.*

1. *If* $\mathbf{u} \in \operatorname{Ker}(A_{G/\pi} - \lambda I) \setminus \{0\}$ *then* $P^\pi \mathbf{u} \in \operatorname{Ker}(A_G - \lambda I) \setminus \{0\}$.
2. *If* $\lambda \in \sigma(A_G)$, $\lambda \notin \sigma(A_{G/\pi})$ *and* $\mathbf{u} \in \operatorname{Ker}(A_G - \lambda I)$ *then* $\mathbf{u}^\top P^\pi = 0$, *i.e., for* $i = 1, 2, \ldots, r$,

$$\sum_{j \in V_i} \mathbf{u}_j = 0.$$

4. Eigenvectors of strongly regular graphs

A *strongly regular graph with parameters* $(n, p; a, c)$ is a p-regular graph $(0 < p < n - 1)$ such that each pair of adjacent vertices has a common neighbors and each pair of nonadjacent vertices has c common neighbors. It is well known that a connected strongly regular graph has diameter 2 and three distinct eigenvalues (note that any graph with diameter d has at least $d + 1$ distinct eigenvalues). One of these eigenvalues is p and the other two, usually called *restricted eigenvalues*, are such that (see, for instance, [6])

$$\lambda = \frac{a - c \pm \sqrt{(a-c)^2 + 4(p-c)}}{2}, \tag{4}$$

The Petersen graph, P, depicted in Fig. 2, is a strongly regular graph with parameters (10, 3; 0, 1) and $\sigma(A_P) = \{[-2]^4, [1]^5, [3]^1\}$.

[1] The exponent denotes the multiplicity of the corresponding eigenvalue between square brackets.

Denoting the distance between the vertices x and y in a graph G by $d_G(x, y)$, it is known [6] that G is distance-regular if and only if the distance partition $V_0, V_1, \ldots, V_d$ is equitable, where for $v \in V(G)$, $V_i = \{x \in V(G) : d_G(v, x) = i\}$, for $i = 1, \ldots, d$, and d is the diameter of G. Therefore, since a strongly regular graph is a distance-regular graph with diameter 2, we may conclude the following well known result (see [6]).

Proposition 4.1. *A graph G is strongly regular with parameters $(n, p; a, c)$ if and only if $\forall k \in V(G)$, there exists the equitable partition $\pi_k = (V_0, V_1, V_2)$, where $V_0 = \{k\}$, $V_1 = N_G(k)$ and $V_2 = V(G) \setminus N_G[k]$, for which*

$$A_{G/\pi_k} = \begin{array}{c} \\ V_0 \\ V_1 \\ V_2 \end{array} \begin{array}{c} \begin{array}{ccc} V_0 & V_1 & V_2 \end{array} \\ \begin{pmatrix} 0 & p & 0 \\ 1 & a & p-a-1 \\ 0 & c & p-c \end{pmatrix} \end{array}. \tag{5}$$

Note that A_G and A_{G/π_k} have the same distinct eigenvalues.

According to [6], a strongly regular graph G is *primitive* if both G and its complement $\bar{G}$ are connected; otherwise it is called *imprimitive*. A strongly regular graph with parameters $(n, p; a, c)$ is imprimitive if and only if $c = p$ or $c = 0$ (see [6]).

In the next result we have a characterization of primitive strongly regular graphs based on the eigenvectors. This characterization will be used in the proof of Proposition 4.3.

Proposition 4.2 [3]. *Let G be a connected graph of order n with at least one edge, $p \in \mathbb{N}$ and $\lambda \in \mathbb{N} \setminus \{p\}$ an eigenvalue of G. Then G is a primitive strongly regular graph with parameters $(n, p; a, c)$ if and only if $\forall k \in V(G)$ $\exists \mathbf{v}^k \in \mathrm{Ker}(A_G - \lambda I) \setminus \{0\}$ such that*

$$\mathbf{v}_i^k = \begin{cases} 1 & \text{if } i = k, \\ \frac{\lambda}{p} & \text{if } i \in N_G(k), \\ \frac{c\lambda}{p(\lambda-(p-c))} & \text{if } i \in V(G) \setminus N_G[k], \end{cases} \tag{6}$$

with $p \neq c \neq 0$.

Proposition 4.3. *Let G be a primitive strongly regular graph with parameters $(n, p; a, c)$ and $\lambda \in \sigma(A_G) \setminus \{p\}$. If there exists $S \subset V(G)$ such that*

$$\forall \mathbf{v} \in \mathrm{Ker}(A_G - \lambda I) \ \sum_{j \in S} \mathbf{v}_j = 0,$$

then S is a (k, τ)-regular set with

$$k = \frac{c\lambda(|S| - 1) + pc}{\lambda(p - \lambda)} - \frac{p}{\lambda} \quad \textit{and} \quad \tau = \frac{c}{p - \lambda}|S|.$$

Proof. Let $S \subset V(G)$ such that $\forall \mathbf{v} \in \mathrm{Ker}(A_G - \lambda I)$ $\sum_{i \in S} \mathbf{v}_i = 0$. Let $x \in V(G)$ and $\mathbf{v}$ be an eigenvector of A_G defined in (6), that is, such that

$$\mathbf{v}_i = \begin{cases} 1 & \text{if } i = x, \\ \frac{\lambda}{p} & \text{if } i \in N_G(x), \\ \frac{c\lambda}{p(\lambda-(p-c))} & \text{if } i \in V(G) \setminus N_G[x], \end{cases}$$

- If $x \in S$, from Proposition 3.3 and from $\lambda \neq \{0, p-c\}$ we have

$$\begin{aligned}0 &= \sum_{j\in S} \mathbf{v}_j \\ &= 1 + \sum_{j\in N_G(x)\cap S} \frac{\lambda}{p} + \sum_{j\in S\setminus N_G[x]} \frac{c\lambda}{p(\lambda-(p-c))} \\ &= \frac{p}{\lambda} + |N_G(x)\cap S| + (|S| - |N_G[x]\cap S|)\frac{c}{\lambda-p+c} \\ &= \frac{p}{\lambda} - \frac{c}{\lambda-p+c} + |N_G(x)\cap S|\frac{\lambda-p}{\lambda-p+c} + |S|\frac{c}{\lambda-p+c} \\ &= |N_G(x)\cap S|(\lambda-p) + \frac{c\lambda(|S|-1)+p(\lambda-p+c)}{\lambda}. \end{aligned} \tag{7}$$

From (7),

$$\begin{aligned}|N_G(x)\cap S| &= \frac{c\lambda(|S|-1)+pc-p(p-\lambda)}{\lambda(p-\lambda)} \\ &= \frac{c\lambda(|S|-1)+pc}{\lambda(p-\lambda)} - \frac{p}{\lambda}.\end{aligned}$$

- If $x \in V(G)\setminus S$ then, again from Proposition 3.3 and from $\lambda \notin \{0, p-c\}$ it is concluded that

$$\begin{aligned}0 &= \sum_{j\in S} \mathbf{v}_j \\ &= \sum_{j\in N_G(x)\cap S} \frac{\lambda}{p} + \sum_{j\in S\setminus N_G(x)} \frac{c\lambda}{p(\lambda-(p-c))} \\ &= |N_G(x)\cap S| + (|S| - |N_G(x)\cap S|)\frac{c}{\lambda-(p-c)} \\ &= |N_G(x)\cap S|\frac{\lambda-p}{\lambda-p+c} + |S|\frac{c}{\lambda-p+c} \\ &= |N_G(x)\cap S|(\lambda-p) + |S|c. \end{aligned} \tag{8}$$

From (8) we get that

$$|N_G(x)\cap S| = \frac{c}{p-\lambda}|S|. \qquad \square$$

Proposition 3.3 combined with Proposition 4.3, allow us to deduce a necessary and sufficient condition for a subset of vertices of a primitive strongly regular graph to be a (k,τ)-regular set.

Corollary 4.1. *Let G be a primitive strongly regular graph with parameters $(n, p; a, c)$, $k \in \mathbb{N}_0$, $\tau \in \mathbb{N}$ and $\lambda \in \sigma(A_G)\setminus\{k-\tau, p\}$. Then $S \subset V(G)$ is (k,τ)-regular if and only if* $\forall \mathbf{v} \in \mathrm{Ker}(A_G - \lambda I)$,

$$\sum_{j\in S} \mathbf{v}_j = 0.$$

In addition, if S is (k,τ)-regular then $k = \frac{c\lambda(|S|-1)+pc}{\lambda(p-\lambda)} - \frac{p}{\lambda}$ and $\tau = \frac{c}{p-\lambda}|S|$.

Considering again the Petersen graph P (depicted in Fig. 2), we may see (by using Matlab, for example) that the characteristic vector of the subset $T = \{1, 4, 7, 8\} \subset V(P)$ is orthogonal to the eigenspace of $1 \in \sigma(A_G)$. Hence, from Proposition 4.3, T is a (k, τ)-regular set of P with

$$k = \frac{1(4-1)+3}{3-1} - \frac{3}{1} = 0 \quad \text{and} \quad \tau = \frac{1}{3-1}4 = 2.$$

Acknowledgments

We are grateful to the referee for a careful reading of the paper, and for its comments and suggestions. The authors were supported by *Centre for Research in Optimization and Control* (*CEOC*) from the "Fundação para a Ciência e a Tecnologia" FCT, cofinanced by the European Community Fund FEDER.

References

[1] D.M. Cardoso, On graphs with stability number equal to the optimal value of a Convex Quadratic Program, Matemática Contemporânea 25 (2003) 9–24.
[2] D.M. Cardoso, P. Rama, Equitable bipartitions of graphs and related results, J. Math. Sci. 120 (2004) 869–880.
[3] D.M. Cardoso, C. Delorme, P. Rama, Laplacian eigenvectors and eigenvalues and almost equitable partitions, Eur. J. Combin., in press, doi:10.1016/j.ejc.2005.03.006.
[4] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, Academic Press, New York, 1979.
[5] D. Cvetković, J.H. van Lint, An elementary proof of Lloyd's theorem, Proc. Kon. Nederl. Acad. Wetensch. Ser. A 80 (1977) 6–10.
[6] C.D. Godsil, Algebraic Combinatorics, Chapman & Hall, New York, 1993.
[7] M.M. Halldórsson, J. Kratochvíl, J.A. Telle, Independent sets with domination constraints, Discrete Appl. Math. 99 (2000) 39–54.
[8] E.V. Haynsworth, Applications of a theorem on partitioned matrices, J. Res. Nat. Bureau Stand. 63 B (1959) 73–78.
[9] G.J. McKay, Backtrack programming and the graph isomorphism problem, M.Sc. thesis, University of Melbourne, 1976.
[10] M. Muzychuk, M. Klin, On graphs with three eigenvalues, Discrete Math. 189 (1998) 191–207.
[11] A. Neumaier, Regular sets and quasi-symmetric 2-designs, in: D. Jungnickel, K. Vedder (Eds.), Combinatorial Theory, Lecture Notes in Math., vol. 969, Springer-Verlag, Berlin, 1982, pp. 258–275.
[12] D.L. Powers, M.M. Sulaiman, The walk partition and colorations of a graph, Linear Algebra Appl. 48 (1982) 145–159.
[13] P. Rama, Propriedades combinatórias e espectrais de grafos com restrições de regularidade, Ph.D. thesis, Universidade de Aveiro, 2005 (in Portuguese).
[14] H. Sachs, Über Teiler, Faktoren und Charakteristische Polynome von Graphen. Teil I, Wiss. Z. TH Ilmenau 12 (1966) 7–12.
[15] H. Sachs, Über Teiler, Faktoren und Charakteristische Polynome von Graphen. Teil II, Wiss. Z. TH Ilmenau 13 (1967) 405–412.
[16] A.J. Schwenk, Computing the characteristic polynomial of a graph, in: R. Bari, F. Harary (Eds.), Graphs and Combinatorics, Springer, Berlin, 1974, pp. 153–172.
[17] P.F. Stadler, G. Tinhofer, Equitable partitions, coherent algebras and random walks: applications to the correlation structure of landscapes, MATCH – Commun. Math. Comput. Chem. 40 (1999) 215–261.
[18] J.A. Telle, Characterization of domination-type parameters in graphs, Congressus Numer. 94 (1993) 9–16.
[19] D.M. Thompson, Eigengraphs: constructing strongly regular graphs with block designs, Utilitas Math. 20 (1981) 83–115.